Theorem 12.1 (Löwenheim-Skolem). *Every infinite model for a countable language has a countable elementary submodel.* □

An *elementary embedding* is an embedding whose range is an elementary submodel.

A set $X \subset A$ is *definable over* $\mathfrak{A}$ if there exist a formula φ and some $a_1, \ldots, a_n \in A$ such that

$$X = \{x \in A : \mathfrak{A} \vDash \varphi[x, a_1, \ldots, a_n]\}.$$

We say that X is *definable in* $\mathfrak{A}$ *from* $a_1, \ldots, a_n$. If φ is a formula of x only, without parameters $a_1, \ldots, a_n$, then X is *definable* in $\mathfrak{A}$. An element $a \in A$ is *definable* (*from* $a_1, \ldots, a_n$) if the set $\{a\}$ is definable (from $a_1, \ldots, a_n$).

Gödel's Theorems

The cornerstone of modern logic are Gödel's theorems: the Completeness Theorem and two incompleteness theorems.

A set Σ of sentences of a language $\mathcal{L}$ is *consistent* if there is no formal proof of contradiction from Σ. The Completeness Theorem states that every consistent set of sentences has a model.

The First Incompleteness Theorem shows that no consistent (recursive) extension of Peano Arithmetic is complete: there exists a statement that is undecidable in the theory. In particular, if ZFC is consistent (as we believe), no additional axioms can prove or refute every sentence in the language of set theory.

The Second Incompleteness Theorem proves that sufficiently strong mathematical theories such as Peano Arithmetic or ZF (if consistent) cannot prove its own consistency. Gödel's Second Incompleteness Theorem implies that it is unprovable in ZF that there exists a model of ZF. This fact is significant for the theory of large cardinals, and we shall return to it later in this chapter.

Direct Limits of Models

An often used construction in model theory is the direct limit of a directed system of models. A *directed set* is a partially ordered set $(D, <)$ such that for every $i, j \in D$ there is a $k \in D$ such that $i \leq k$ and $j \leq k$.

First consider a system of models $\{\mathfrak{A}_i : i \in D\}$, indexed by a directed set D, such that for all $i, j \in D$, if $i < j$ then $\mathfrak{A}_i \prec \mathfrak{A}_j$. Let $\mathfrak{A} = \bigcup_{i \in D} \mathfrak{A}_i$; i.e., the universe of $\mathfrak{A}$ is the union of the universes of the $\mathfrak{A}_i$, $P^{\mathfrak{A}} = \bigcup_{i \in D} P^{\mathfrak{A}_i}$, etc. It is easily proved by induction on the complexity of formulas that $\mathfrak{A}_i \prec \mathfrak{A}$ for all i.

In general, we consider a *directed system* of models which consists of models $\{\mathfrak{A}_i : i \in D\}$ together with elementary embeddings $e_{i,j} : \mathfrak{A}_i \to \mathfrak{A}_j$ such that $e_{i,k} = e_{j,k} \circ e_{i,j}$ for all $i < j < k$.